Comment on a theorem of Hojman and its generalizations

1996 J. Phys. A: Math. Gen. 29 6999

(http://iopscience.iop.org/0305-4470/29/21/030)

J. Phys. A: Math. Gen. **29** (1996) 6999–7002. Printed in the UK

COMMENT

Comment on a theorem of Hojman and its generalizations

T Pillay and P G L Leach†

Department of Mathematics and Applied Mathematics, University of Natal, Private Bag X10, Dalbridge 4014, South Africa

Received 1 August 1996

Abstract. An invariant derived using Hojman's conservation law is shown to be trivial for all Noether symmetries.

The traditional approaches to the problem of finding conserved quantities of systems of equations may be divided into those which use symmetry methods, such as the Lie or Noether approach, and those which do not, such as the direct method. This division is somewhat artifical since all of these methods share an underlying symmetry basis. Still, it remains true that this is more explicit in some cases than in others.

Our concern in this comment lies with those methods which use symmetries explicitly. A new conservation law based solely on the existence of symmetries has been presented by Hojman [1], and subsequently generalized by González-Gascón [2] and further by Lutzky [3]. Lutzky uses this theorem to derive an invariant for the case where a Lagrangian does exist. We demonstrate that this invariant is trivial if the symmetry used is a Noether symmetry. This has important practical implications for the use of this theorem.

We begin by briefly outlining the use of symmetry methods to find invariants. Neither the Lie method nor the use of Noether's theorem provides a definitive answer to this problem. Noether's theorem has the advantage that once one has found a (Noether) symmetry the first integral follows easily. The Lie approach generally yields more symmetries and hence the potential for more first integrals. For an nth order scalar equation the Noether approach produces one integral per symmetry. The Lie approach, on the other hand, produces $(n-1)$ integrals per symmetry. However, the calculation of these first integrals is decidedly nontrivial [4].

The common area for the Lie and Noether approaches is found in point symmetries. Moving away from this we obtain, in the Lie case, contact symmetries. In the Noether case we have no such restriction. We can obtain velocity-dependent symmetries of any order. Thus higher-order Lagrangians give rise to higher-order Noether symmetries, but the application of the Lie theory to the corresponding equations of motion is restricted to point or contact symmetries.

The most general form [3] of the conservation theorem first presented by Hojman [1] may be stated as follows. Let the functions $\eta_l(x, y, y')$ determine a symmetry generator

† Member of the Centre for Theoretical and Computational Chemistry, University of Natal, Durban, and Associate Member of the Centre for Nonlinear Studies, University of the Witwatersrand, Johannesburg.

0305-4470/96/216999+04$19.50

$G = \eta_l \partial/\partial y_l + \eta_l' \partial/\partial y_l'$ for the equations of motion $y_l'' = \alpha_l(x, y, y')$; if a function $\lambda(x, y, y')$ can be found such that the quantity

$$\Omega = \frac{\partial \alpha_l}{\partial y_l'} + \frac{\mathrm{d}}{\mathrm{d}x} \log \lambda \tag{1}$$

is an invariant of the symmetry group, then a constant of the motion is given by

$$\phi = \frac{1}{\lambda} \frac{\partial}{\partial y}(\lambda n) + \frac{1}{\lambda} \frac{\partial}{\partial y'}(\lambda \eta'). \tag{2}$$

Note that the statement of the theorem immediately excludes contact symmetries since these require that the coefficient of $\partial/\partial x$ in the generator be nonzero. Further, Hojman shows that if the transformation η is a point transformation then the invariant is trivial.

We now show that this invariant is trivial if η represents a Noether transformation. The proof is for the one-dimensional case, but it generalizes easily to higher dimensions. If a Lagrangian exists (as we require to apply Noether's theorem), then it can be shown quite generally that [5]

$$\frac{\partial \alpha_l}{\partial y_l'} + \frac{\mathrm{d}}{\mathrm{d}x}(\log D) = 0 \tag{3}$$

where D is the Hessian of L with respect to the velocities. We may choose $\lambda = D$ and so $\Omega = 0$ which satisfies the condition of Hojman's theorem. The condition for the $\eta \partial/\partial y + \eta' \partial/\partial y'$ to be a Noether symmetry is [6]

$$\eta \frac{\partial L}{\partial y} + \eta' \frac{\partial L}{\partial y'} = f'(x, y, y') \tag{4}$$

where L is the Lagrangian and f an arbitrary function. Separating out y'' terms in (4) we obtain the following conditions

$$\eta \frac{\partial L}{\partial y} + \left(\frac{\partial \eta}{\partial x} + y' \frac{\partial \eta}{\partial y} \right) \frac{\partial L}{\partial y'} = \frac{\partial f}{\partial x} + y' \frac{\partial f}{\partial y} \tag{5}$$

$$y'' \frac{\partial \eta}{\partial y'} \frac{\partial L}{\partial y'} = y'' \frac{\partial f}{\partial y'}. \tag{6}$$

From (6)

$$f = \eta \frac{\partial L}{\partial y'} - \int \eta \frac{\partial^2 L}{\partial y'^2} \, \mathrm{d}y' + c(x, y).$$

Let

$$\nu = \int \eta \frac{\partial^2 L}{\partial y'^2} \, \mathrm{d}y'. \tag{7}$$

Differentiation of (7) with respect to y' gives

$$\eta = \frac{1}{\lambda} \frac{\partial \nu}{\partial y'} \tag{8}$$

where $\lambda = D = \partial^2 L/\partial y'^2$. Substitution for η in (5) gives, after some simplification, the following equation for ν:

$$\frac{\partial \nu}{\partial x} + y' \frac{\partial \nu}{\partial y} + \frac{\partial \nu}{\partial y'} \left(\frac{1}{\lambda} \frac{\partial L}{\partial y} - \frac{1}{\lambda} \frac{\partial^2 L}{\partial x \partial y'} - \frac{y'}{\lambda} \frac{\partial^2 L}{\partial y \partial y'} \right) = 0. \tag{9}$$

The Euler–Lagrange equation gives

$$\begin{aligned}\frac{\partial L}{\partial y} &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial y'} \\ &= \frac{\partial^2 L}{\partial x \partial y'} + y'\frac{\partial^2 L}{\partial y \partial y'} + y''\frac{\partial^2 L}{\partial y'^2} \\ &= \frac{\partial^2 L}{\partial x \partial y'} + y'\frac{\partial^2 L}{\partial y \partial y'} + y''\lambda. \end{aligned} \tag{10}$$

Substituting (10) in (9) and simplifying we obtain

$$\frac{\partial \nu}{\partial x} + y'\frac{\partial \nu}{\partial y} + y''\frac{\partial \nu}{\partial y'} = 0$$

that is,

$$\nu' = 0. \tag{11}$$

From (8)

$$\lambda\eta = \frac{\partial \nu}{\partial y'}. \tag{12}$$

Taking total derivatives on both sides of (12), and making use of (11) and the identity

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial}{\partial y'}A - \frac{\partial}{\partial y'}\frac{\mathrm{d}}{\mathrm{d}x}A = -\frac{\partial A}{\partial y} - \frac{\partial \alpha}{\partial y'}\frac{\partial A}{\partial y'}$$

for all functions $A(x, y, y')$ [1], we obtain

$$\begin{aligned}\lambda'\eta + \lambda\eta' &= -\frac{\partial \nu}{\partial y} - \frac{\partial \nu}{\partial y'}\frac{\partial \alpha}{\partial y'} \\ \lambda\eta' &= -\frac{\partial \nu}{\partial y} - \frac{\partial \nu}{\partial y'}\frac{\partial \alpha}{\partial y'} - \lambda'\eta \\ &= -\frac{\partial \nu}{\partial y} - \frac{\partial \nu}{\partial y'}\frac{\partial \alpha}{\partial y'} - \frac{\lambda'}{\lambda}\frac{\partial \nu}{\partial y'}.\end{aligned}$$

From (1), with $\Omega = 0$ we have that

$$\frac{\lambda'}{\lambda} = -\frac{\partial \alpha}{\partial y'}$$

and so

$$\lambda\eta' = -\frac{\partial \nu}{\partial y}. \tag{13}$$

Substituting (12) and (13) into (2) we find that the invariant is

$$\begin{aligned}\phi &= \frac{1}{\lambda}\frac{\partial^2 \nu}{\partial y \partial y'} - \frac{1}{\lambda}\frac{\partial^2 \nu}{\partial y' \partial y} \\ &= 0.\end{aligned} \tag{14}$$

Thus this theorem gives trivial results for point, contact and Noether transformations. It may still be useful if one is working with (non-Noether) generalized symmetries. However, this leaves one with the problem of finding such generalized symmetries.

Acknowledgments

TP and PGLL thank the Foundation for Research Development of South Africa and the University of Natal for their continuing support.

References

[1] Hojman S A 1992 A new conservation law constructed without using either Lagrangians or Hamiltonians *J. Phys. A: Math. Gen.* **25** L291–5
[2] González-Gascón F 1994 Geometric foundations of a new conservation law discovered by Hojman *J. Phys. A: Math. Gen.* **27** L59–60
[3] Lutzky M 1995 Remarks on a recent theorem about conserved quantities *J. Phys. A: Math. Gen.* **28** L637–8
[4] Gorringe V M and Leach P G L 1991 First integrals associated with the additional symmetry of central force problems with power law potentials *Quæstiones Mathematicæ* **14** 277–89
[5] Lutzky M 1979 Origin of nonNoether invariants *Phys. Lett.* **75A** 8–10
[6] Sarlet W and Cantrijn F 1981 Generalizations of Noether's theorem in classical mechanics *SIAM Rev.* **23** 467–94